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Step algebras of quantum algebras of type A, B and D

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Abstract. We study the step algebras for the quantum algebras corresponding to the Lie algebra pairs $A_{n-1} \subset A_n$, $B_{n-1} \subset B_n$ and $D_{n-1} \subset D_n$.

1. Introduction

In [14] Mickelsson introduced the step algebra $S(\mathfrak{k}, \mathfrak{g})$ of the Lie algebra pair $(\mathfrak{k}, \mathfrak{g})$, where $\mathfrak{k}$ is a semisimple Lie subalgebra of a finite-dimensional Lie algebra $\mathfrak{g}$. The theory of step algebras was later also developed by van den Hombergh [3] and Zhelobenko [19, 20]. Step algebra methods have been applied to the representation theory of semisimple Lie algebras [11, 5], Lie superalgebras [12] and Kac–Moody algebras [13].

A step algebra is defined by $S(\mathfrak{g}, \mathfrak{k}) = S'(\mathfrak{g}, \mathfrak{k})/U(\mathfrak{g})\mathfrak{k}_+$, where $S'(\mathfrak{g}, \mathfrak{k}) = \{u \in U(\mathfrak{g}) | \mathfrak{k}_+ u \subset U(\mathfrak{g})\mathfrak{k}_+\}$. Step algebras are useful for studying irreducible $\mathfrak{k}$-finite $\mathfrak{g}$-modules. $\mathfrak{g}$-module V is $\mathfrak{k}$-finite, if it is a direct sum of finite-dimensional $\mathfrak{k}$-modules. Let V^+ be the set of $\mathfrak{k}$-maximal vectors in V. Step algebra $S(\mathfrak{g}, \mathfrak{k})$ operates in V^+ in a natural way. One basic result in the theory of step algebras is the existence of a certain subalgebra $S_0(\mathfrak{g}, \mathfrak{k}) \subset S(\mathfrak{g}, \mathfrak{k})$, which generates the whole V^+ from a single $v \in V^+$.

In this paper we study the step algebras of q-deformations of the following Lie algebra pairs: $A_{n-1} \subset A_n$, $B_{n-1} \subset B_n$ and $D_{n-1} \subset D_n$. As the main result we prove the existence of the subalgebra $S_0(\mathfrak{g}, \mathfrak{k})$ in these q-deformed cases. Earlier in [6] we studied the step algebra of the q-deformation of the Lie algebra pair $\mathfrak{sl}(n-1) \subset \mathfrak{sl}(n)$.

2. Quantum algebra $U_q(\mathfrak{g})$

Let $\mathfrak{g}$ be a simple complex Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra and $\{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^*$ a basis of simple roots.

Let $(\cdot|\cdot)$ be the dual of the Killing form restricted to $\mathfrak{h}$ and $(a_{ij})_{i,j=1}^n$, $a_{ij} = \langle \alpha_i | \alpha_j \rangle = 2(\alpha_i|\alpha_j)/(\alpha_j|\alpha_j)$, the Cartan matrix of $\mathfrak{g}$.

For $q \in \mathbb{C}^*$ let

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad [n]_q! = [1]_q[2]_q \cdots [n]_q \qquad [0]_q! = 1$$
$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[m-n]_q![n]_q!}.$$

$U_q(\mathfrak{g})$ is the associative algebra over $\mathbb{C}$ generated by the elements $k_i^{\pm 1}$, e_i and f_i, $i = 1, \dots, n$, with relations

$$k_i k_i^{-1} = k_i^{-1} k_i = 1 \qquad k_i k_j = k_j k_i$$

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$$k_i e_j k_i^{-1} = q_i^{a_{ij}} e_j \qquad k_i f_j k_i^{-1} = q_i^{-a_{ij}} f_j$$
$$[e_i, f_j] = \delta_{ij} \frac{k_i^2 - k_i^{-2}}{q_i^2 - q_i^{-2}}$$
$$\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \begin{bmatrix} 1-a_{ij} \\ \nu \end{bmatrix}_{q_i^2} e_i^{1-a_{ij}-\nu} e_j e_i^\nu = 0 \qquad i \neq j$$
$$\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \begin{bmatrix} 1-a_{ij} \\ \nu \end{bmatrix}_{q_i^2} f_i^{1-a_{ij}-\nu} f_j f_i^\nu = 0 \qquad i \neq j$$

where $q_i = q^{(\alpha_i|\alpha_i)/2}$, so $q_i^{a_{ij}} = q_j^{a_{ji}} = q^{(\alpha_i|\alpha_j)}$. Quantum algebras were introduced by Drinfeld [2] and Jimbo [4].

$U_q(\mathfrak{g})$ is a Hopf algebra with a coproduct $\Delta : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ defined by

$$\Delta(k_i^{\pm 1}) = k_i^{\pm 1} \otimes k_i^{\pm 1} \qquad \Delta(e_i) = e_i \otimes 1 + k_i^{-2} \otimes e_i \qquad \Delta(f_i) = f_i \otimes k_i^2 + 1 \otimes f_i$$

and a counit $\epsilon : U_q(\mathfrak{g}) \to \mathbb{C}$ and an antipode $S : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ defined by

$$\epsilon(k_i) = \epsilon(k_i^{-1}) = 1 \qquad \epsilon(e_i) = \epsilon(f_i) = 0$$
$$S(k_i) = k_i^{-1} \qquad S(e_i) = -k_i^2 e_i \qquad S(f_i) = -f_i k_i^{-2}.$$

Recall that Δ and ϵ (respectively S) extend to an algebra (anti) homomorphism of $U_q(\mathfrak{g})$.

Let L (respectively R) be the left (respectively right) regular representation of $U_q(\mathfrak{g})$. The adjoint representation of $U_q(\mathfrak{g})$ is defined by $\mathrm{ad} = (L \otimes R)(\mathrm{Id} \otimes S)\Delta$.

From now on we assume that q is not a root of unity.

Notations.

Let $U_q(\mathfrak{g}_+) \subset U_q(\mathfrak{g})$ (respectively $U_q(\mathfrak{g}_-)$) be the subalgebra generated by the elements e_i (respectively f_i), $i = 1, \ldots, n$ and $U_q(\mathfrak{h})$ the subalgebra generated by the elements $k_i^{\pm 1}$. Then $U_q(\mathfrak{h})$ is the algebra of Laurent polynomials in the indeterminates k_i.

For $\alpha = \sum_{i=1}^n l_i \alpha_i$, $l_i \in \mathbb{Z}$, set $k_\alpha = k_1^{l_1} \ldots k_n^{l_n}$.

2.1. Verma modules

Let V be a $U_q(\mathfrak{g})$-module (see [9, 17]). A vector $0 \neq v \in V$ is a weight vector of weight $\omega = (\omega_1, \ldots, \omega_n) \in (\mathbb{C}^*)^n$, if $k_i v = \omega_i v$ for all $i = 1, \ldots, n$. Such a ω generates a character $\omega : U_q(\mathfrak{h}) \to \mathbb{C}$. Now

$$V_\omega = \{v \in V | k_i v = \omega_i v, \ i = 1, \ldots, n\}$$

is the weight subspace corresponding to the weight ω. A vector $0 \neq v \in V_\omega$ is a highest weight vector if $e_i v = 0$ for all $i = 1, \ldots, n$. V is a highest weight module, if it is generated by a highest weight vector.

As in the theory of semisimple Lie algebras for each $\omega \in (\mathbb{C}^*)^n$ we can construct the Verma module $M(\omega)$ and the unique irreducible standard cyclic module $V(\omega)$ with highest weight ω.

For each $\lambda \in \mathfrak{h}^*$ we can construct a character $q^\lambda : U_q(\mathfrak{h}) \to \mathbb{C}$ by defining $q^\lambda(k_i) = q^{(\lambda|\alpha_i)}$. If V is an irreducible finite-dimensional $U_q(\mathfrak{g})$-module, then it is a highest weight module with the highest weight $\omega = \epsilon \cdot q^\lambda = (\epsilon_1 q^{(\lambda|\alpha_1)}, \ldots, \epsilon_n q^{(\lambda|\alpha_n)})$, where $\lambda \in \Lambda^+$ is a dominant integral weight of $\mathfrak{g}$, and $\epsilon_i^4 = 1$. We denote by Ω^+ the set of weights of this form. For any weight $\omega \in \Omega^+$ $V(\omega)$ is finite-dimensional. Furthermore $V(\epsilon \cdot q^\lambda) \cong V(q^\lambda) \otimes \mathbb{C}_\epsilon$, where $\mathbb{C}_\epsilon$ is a one-dimensional $U_q(\mathfrak{g})$-module with the highest weight ϵ. Since $U_q(\mathfrak{g})$ is a Hopf algebra, we can (using Δ) define a $U_q(\mathfrak{g})$-module structure on the tensor product of $U_q(\mathfrak{g})$-modules.

2.2. The Harish–Chandra theorem

Let us denote by $Z(\mathfrak{g})$ the centre of $U_q(\mathfrak{g})$. Let $v \in M(q^\lambda)$ be a highest weight vector. Then for all $z \in Z(\mathfrak{g})$, zv is a highest weight vector with weight q^λ, so $zv = \chi_\lambda(z)v$, where $\chi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$ is an algebra homomorphism.

Let δ be the half sum of the positive roots of $\mathfrak{g}$ and $W_\mathfrak{g}$ the Weyl group of $\mathfrak{g}$. Let $\sim$ be the equivalence relation in $\mathfrak{h}^*$ defined by: $\lambda \sim \mu \Leftrightarrow \exists w \in W_\mathfrak{g}$ such that $\lambda + \delta = w(\mu + \delta)$.

Now we can formulate the analogue of the Harish–Chandra theorem, see [15, 18].

Harish–Chandra theorem. If $\chi_\lambda = \chi_\mu$, then $\lambda \sim \mu$.

2.3. The Cartan–Weyl basis and the Poincaré–Birkhoff–Witt theorem

The system of positive roots $\Delta_+^\mathfrak{g}$ of $\mathfrak{g}$ is normally ordered if each root which is a sum of other roots lies between its summands, and $\alpha < \beta$ if α is before β in the normal ordering.

The q-analogue of the Cartan–Weyl basis is constructed inductively in the following way [7]. Fix some normal ordering in $\Delta_+^\mathfrak{g}$. For a simple root α_i we define $e_{\alpha_i} = e_i$. For a non-simple root $\gamma = \alpha + \beta$, where $\alpha, \beta \in \Delta_+^\mathfrak{g}$ and $\alpha < \beta$ such that there are no other roots α' and β' between α and β for which $\gamma = \alpha' + \beta'$, we set

$$e_\gamma = [e_\alpha, e_\beta]_{q^2} \qquad e_{-\gamma} = [e_{-\beta}, e_{-\alpha}]_{q^{-2}}$$

if e_α and e_β are already constructed. Here $[e_\alpha, e_\beta]_q = e_\alpha e_\beta - q^{-(\alpha,\beta)} e_\beta e_\alpha$.

The q-analogues of the Cartan–Weyl generators satisfy commutation relations of the following form. For any $\gamma \in \Delta_+^\mathfrak{g}$

$$[e_\gamma, e_{-\gamma}] = a_\gamma(q) \frac{k_\gamma^2 - k_{-\gamma}^2}{q^2 - q^{-2}}.$$

For $\alpha, \beta \in \Delta_+^\mathfrak{g}$, $\alpha < \beta$

$$[e_\alpha, e_\beta]_{q^2} = \sum_{\alpha < \nu_1 < \cdots < \nu_m < \beta} b_{l_i, \nu_i}(q) e_{\nu_1}^{l_1} e_{\nu_2}^{l_2} \ldots e_{\nu_m}^{l_m}$$

where $\sum_i l_i \nu_i = \alpha + \beta$, and

$$[e_\beta, e_{-\alpha}] = \sum_{\substack{\nu_1 < \cdots < \nu_m < \alpha \\ \beta < \nu_1' < \cdots < \nu_r'}} c_{l_i, \nu_i, l_i', \nu_i'}(q, k_\alpha, k_\beta) e_{-\nu_m}^{l_m} \ldots e_{-\nu_1}^{l_1} e_{\nu_1'}^{l_1'} \ldots e_{\nu_r'}^{l_r'}$$

where $\sum_i (l_i' \nu_i' - l_i \nu_i) = \beta - \alpha$. The explicit form of the coefficients a, b and c in the rank two case can be found in [7].

Let $\Delta_+^\mathfrak{g} = \{\gamma_1, \ldots, \gamma_s\}$ be normally ordered. The monomials $e_{-\gamma_s}^{n_s} \ldots e_{-\gamma_1}^{n_1} e_{\gamma_1}^{m_1} \ldots e_{\gamma_s}^{m_s}$ $k_1^{l_1} \ldots k_n^{l_n}$; $n_i, m_i \in \mathbb{N}$, $k_i \in \mathbb{Z}$ form a Poincaré–Birkhoff–Witt basis for $U_q(\mathfrak{g})$, see [8, 10, 16].

3. Step algebra $S_q(\mathfrak{g}, \mathfrak{k})$

3.1. Cartan–Weyl generators

Let $\mathfrak{g}$ be a Lie algebra of type A_n, B_n or D_n and $U_q(\mathfrak{k}) \subset U_q(\mathfrak{g})$ a subalgebra generated by the elements $k_i^{\pm 1}$, e_i and f_i, $i = 2, \ldots, n$. Then $U_q(\mathfrak{k})$ is a q-deformation of Lie algebra $\mathfrak{k}$ of the type A_{n-1}, B_{n-1} and D_{n-1}, respectively.

Next we fix the normal ordering of the positive roots and Cartan–Weyl generators for these algebras. Let $\{\alpha_1, \ldots, \alpha_n\}$ be a set of simple roots of $\mathfrak{g}$; the simple roots of subalgebra $\mathfrak{k} \subset \mathfrak{g}$ are $\{\alpha_2, \ldots, \alpha_n\}$.

In the case $\mathfrak{g} = A_n$ the positive roots are

$$\sum_{j=l}^{k} \alpha_j \qquad l \leqslant k.$$

We fix the normal ordering inductively by setting

$$A_2 : \{\alpha_1, \alpha_1 + \alpha_2, \alpha_2\}$$
$$A_n : \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_n, \text{ normally ordered } A_{n-1}\}.$$

Cartan–Weyl generators are defined by

$$e_{\alpha_l+\cdots+\alpha_{k+1}} = [e_{\alpha_l+\cdots+\alpha_k}, e_{\alpha_{k+1}}]_{q^2}.$$

The positive roots of $\mathfrak{g} = B_n$ are

$$\sum_{j=l}^{k-1} \alpha_j \qquad l < k \leqslant n+1$$
$$\sum_{j=l}^{k-1} \alpha_j + 2\sum_{j=k}^{n} \alpha_j \qquad l < k \leqslant n$$

and the normal ordering is defined as follows:

$$B_2 : \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_2\}$$
$$B_n : \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_n, \alpha_1 + \cdots + \alpha_{n-1} + 2\alpha_n, \ldots, \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n, \text{ normally ordered } B_{n-1}\}$$

and the Cartan–Weyl elements are

$$e_{\alpha_l+\cdots+\alpha_{k+1}} = [e_{\alpha_l+\cdots+\alpha_k}, e_{\alpha_{k+1}}]_{q^2}$$
$$e_{\alpha_l+\cdots+\alpha_{n-1}+2\alpha_n} = [e_{\alpha_l+\cdots+\alpha_{n-1}+\alpha_n}, e_{\alpha_n}]_{q^2}$$
$$e_{\alpha_l+\cdots+\alpha_{k-1}+2\alpha_k+\cdots+2\alpha_n} = [e_{\alpha_l+\cdots+\alpha_k+2\alpha_{k+1}+\cdots+2\alpha_n}, e_{\alpha_k}]_{q^2}.$$

$\mathfrak{g} = D_n$ has the positive roots

$$\sum_{j=l}^{k-1} \alpha_j \qquad l < k \leqslant n \text{ or } l < n-1,\ k = n+1$$
$$\sum_{j=l}^{k-1} \alpha_j + 2\sum_{j=k}^{n-2} \alpha_j + \alpha_{n-1} + \alpha_n \qquad l < k \leqslant n-2$$
$$\sum_{j=l}^{n-2} \alpha_j + \alpha_n \qquad l \leqslant n-2, \text{ and } \alpha_n.$$

The normal ordering is

$$D_4 : \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4, \alpha_2, \alpha_2 + \alpha_3, \alpha_3, \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_4, \alpha_4\}$$
$$D_n : \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{n-2} + \alpha_{n-1}, \alpha_1 + \cdots + \alpha_{n-2} + \alpha_n, \alpha_1 + \cdots + \alpha_{n-1} + \alpha_n, \alpha_1 + \cdots + \alpha_{n-3} + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n, \ldots, \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n, \text{ normally ordered } D_{n-1}\}$$

and the Cartan–Weyl elements are

$$
\begin{aligned}
&e_{\alpha_l+\cdots+\alpha_{k+1}} = [e_{\alpha_l+\cdots+\alpha_k}, e_{\alpha_{k+1}}]_{q^2} \qquad k < n-1\\
&e_{\alpha_l+\cdots+\alpha_{n-2}+\alpha_n} = [e_{\alpha_l+\cdots+\alpha_{n-2}}, e_{\alpha_n}]_{q^2}\\
&e_{\alpha_l+\cdots+\alpha_{n-1}+\alpha_n} = [e_{\alpha_l+\cdots+\alpha_{n-2}+\alpha_n}, e_{\alpha_{n-1}}]_{q^2} \qquad l < n-2\\
&e_{\alpha_{n-2}+\alpha_{n-1}+\alpha_n} = [e_{\alpha_{n-1}}, e_{\alpha_{n-2}+\alpha_n}]_{q^2}\\
&e_{\alpha_l+\cdots+\alpha_{n-3}+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n} = [e_{\alpha_l+\alpha_{n-2}+\alpha_{n-1}+\alpha_n}, e_{\alpha_{n-2}}]_{q^2}\\
&e_{\alpha_l+\cdots+\alpha_{k-1}+2\alpha_k+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n} = [e_{\alpha_l+\cdots+\alpha_k+2\alpha_{k+1}+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n}, e_{\alpha_k}]_{q^2}.
\end{aligned}
$$

3.2. *Step algebra* $S_q(\mathfrak{g}, \mathfrak{k})$

From now on we denote by $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{k}$, and so $U_q(\mathfrak{h})$ is generated by the elements $k_i^{\pm 1}$, $i = 2, \ldots, n$. Let $\mathfrak{k}_+ \subset U_q(\mathfrak{g})$ be the vector space spanned by $\{e_\alpha | \alpha \in \Delta_+^{\mathfrak{k}}\}$. We define

$$S_q'(\mathfrak{g}, \mathfrak{k}) = \{u \in U_q(\mathfrak{g}) | \mathfrak{k}_+ u \subset U_q(\mathfrak{g})\mathfrak{k}_+\}$$

and the step algebra

$$S_q(\mathfrak{g}, \mathfrak{k}) = S_q'(\mathfrak{g}, \mathfrak{k}) / U_q(\mathfrak{g})\mathfrak{k}_+.$$

$S_q(\mathfrak{g}, \mathfrak{k})$ is an adjoint $U_q(\mathfrak{h})$-module and it is a direct sum of weight subspaces $S_q(\mathfrak{g}, \mathfrak{k})_{q^\mu}$.

Using the Poincaré–Birkhoff–Witt theorem we may split

$$U_q(\mathfrak{g}) = U_1 U_q(\mathfrak{h}) \oplus U_q(\mathfrak{k}_-)\mathfrak{k}_- U_1 U_q(\mathfrak{h}) \oplus U_q(\mathfrak{g})\mathfrak{k}_+ \tag{1}$$

where $U_1 \subset U_q(\mathfrak{g})$ is the vector space spanned by the monomials $e_\gamma^{\bar{n}} = e_{-\gamma_p}^{n_p} \ldots e_{-\gamma_1}^{n_1} e_0^{n_0} e_{\gamma_1}^{n_1'} \ldots e_{\gamma_p}^{n_p'}$ where $\Delta_+^{\mathfrak{g}} = \{\gamma_1, \ldots, \gamma_s\}$ is normally ordered and $\Delta_+^{\mathfrak{g}} \backslash \Delta_+^{\mathfrak{k}} = \{\gamma_1, \ldots, \gamma_p\}$ and $e_0 = k_1$; $n_i, n_i' \in \mathbb{N}$, $n_0 \in \mathbb{Z}$.

Let $I_\omega \subset U_q(\mathfrak{g})$ be the left ideal generated by $\mathfrak{k}_+$ and the elements $k_i - \omega_i \cdot \mathbf{1}$, $i = 2, \ldots, n$ and

$$N^\omega = U_q(\mathfrak{g}) / I_\omega.$$

N^ω is a left $U_q(\mathfrak{k})$-module in a natural way. Furthermore, for $u \in U_q(\mathfrak{h})$ we define $u(\omega) \in \mathbb{C}$ by $u \equiv u(\omega) \cdot \mathbf{1} \bmod I_\omega$. Then $u(\omega)$ is a Laurent polynomial in the variables ω_i; it is obtained via the replacement $k_i \mapsto \omega_i$ in u.

Let $P' : U_q(\mathfrak{g}) \to U_1 U_q(\mathfrak{h})$ be the projection on the first summand in (1) and define $P : S_q(\mathfrak{g}, \mathfrak{k}) \to U_1 U_q(\mathfrak{h})$ by $P(s + U_q(\mathfrak{g})\mathfrak{k}_+) = P'(s)$.

Theorem 1. The mapping $P : S_q(\mathfrak{g}, \mathfrak{k}) \to U_1 U_q(\mathfrak{h})$ is injective.

Proof. Let $s \in S_q(\mathfrak{g}, \mathfrak{k})$ be such that $P(s) = 0$. We may assume that s has weight q^μ. Now

$$s = \sum_{\bar{m}\gamma \gg \mu} v_{\bar{m}} e_\gamma^{\bar{m}} u_{\bar{m}}$$

where $\bar{m}\gamma = (m_1' - m_1)\gamma_1 + \cdots + (m_p' - m_p)\gamma_p$ and $\gg$ is the order defined by the simple roots of $\mathfrak{k}$ and $v_{\bar{m}} \in U_q(\mathfrak{k}_-)$, $u_{\bar{m}} \in U_q(\mathfrak{h})$.

If $s \neq 0$, then we choose a weight $\lambda \in \mathfrak{h}^*$ such that $\lambda + \mu \in \Lambda^+$ and $u_{\bar{m}}(q^\lambda) \neq 0$ for some $\bar{m}$; let $\bar{m}_o$ be the one for which $\bar{m}_o\gamma$ is minimal. Because $e_\alpha s \equiv 0 \bmod U_q(\mathfrak{g})\mathfrak{k}_+$ for all $\alpha \in \Delta_+^{\mathfrak{k}}$, then $v_{\bar{m}_o}$ is a highest weight vector of the weight $q^{\lambda+\mu}$ in the $U_q(\mathfrak{k})$ Verma module $M(q^{\lambda + \bar{m}_o\gamma})$. On the other hand $\lambda + \mu \ll \lambda + \bar{m}_o\gamma$, a contradiction. So $s = 0$. □

Let $\{\gamma \mid \pm\gamma \in \Delta_+^{\mathfrak{g}} \backslash \Delta_+^{\mathfrak{k}}$ or $\gamma = 0\} = \{\mu_1, \ldots, \mu_{2p+1}\}$ be ordered weight monotonically i.e. if $\mu_i \gg \mu_j$, then $i > j$. Furthermore, let $\Delta_+^{\mathfrak{k}} = \{\beta_1, \ldots, \beta_r\}$ (normally ordered).

Theorem 2. For each e_{μ_i}, $i = 1, \ldots, 2p+1$, there exists $s_{\mu_i} \in S_q(\mathfrak{g}, \mathfrak{k})$ such that s_{μ_i} has a weight q^{μ_i} and

$$P(s_{\mu_i}) = e_{\mu_i} u_i$$

where $u_i \in U_q(\mathfrak{h})$ such that $u_i(q^\lambda) \neq 0$ if λ satisfies the following condition:

($*$) there exists no $w \in W_{\mathfrak{k}}$ such that $w(\lambda + \mu_i + \delta) = \lambda + \mu_j + \delta$ for some $j > i$.

Furthermore, if $s \in S_q(\mathfrak{g}, \mathfrak{k})$ such that s has weight q^{μ_i} and $P(s) = e_{\mu_i} u$, $u \in U_q(\mathfrak{h})$, then $s \in U_q(\mathfrak{h}) s_{\mu_i}$.

Remark. If $\lambda + \mu_i \in \Lambda^+$ then λ satisfies condition ($*$).

Proof. We will first prove that there exist unique elements $u_{ij} \in U_q(\mathfrak{k}_-)$ such that $e_{\mu_i} + \sum_{j>i} u_{ij} e_{\mu_j} \in N^{q^\lambda}$ is a highest weight vector with the weight $q^{\lambda+\mu_i}$.

Let $L_j^\lambda \subset N^{q^\lambda}$ be the left $U_q(\mathfrak{k})$-module generated by the elements e_{μ_k} with $k \geqslant j$. Using the Poincaré–Birkhoff–Witt theorem and the commutation relations of the Cartan–Weyl elements, we see that

$$L_j^\lambda = \sum_{k \geqslant j} U_q(\mathfrak{k}_-) e_{\mu_k}$$

and L_j^λ is a free $U_q(\mathfrak{k}_-)$-module with basis $\{e_{\mu_k} | k \geqslant j\}$.

Now $\{0\} = L_{2p+2}^\lambda \subset L_{2p+1}^\lambda \subset \cdots \subset L_i^\lambda$ and the left $U_q(\mathfrak{k})$-module $L_j^\lambda / L_{j+1}^\lambda$ is equivalent with Verma module $M(q^{\lambda+\mu_j})$ with a highest weight vector $e_{\mu_j} + L_{j+1}^\lambda$.

Let Z_j^λ be the kernel of the character $\chi_{\lambda+\mu_j} : Z(\mathfrak{k}) \to \mathbb{C}$. $Z_j^\lambda \subset Z(\mathfrak{k})$ is a maximal ideal. If λ satisfies condition ($*$) then by the Harish–Chandra theorem $Z_i^\lambda \neq Z_j^\lambda$ for $j = i+1, \ldots, 2p+1$; so Z_i^λ and Z_j^λ are relatively prime. Then by proposition II 1.4. in [1] the ideals Z_i^λ and $\prod_{j=i+1}^{2p+1} Z_j^\lambda$ are relatively prime; so there exist $a \in Z_i^\lambda$ and $b \in \prod_{j=i+1}^{p} Z_j^\lambda$ such that $\mathbf{1} = a + b$.

Since $bL_{i+1}^\lambda = \{0\}$, then $be_{\mu_i} \in L_i^\lambda$ is a highest weight vector with the weight $q^{\lambda+\mu_i}$. On the other hand $be_{\mu_i} = e_{\mu_i} - ae_{\mu_i} \equiv e_{\mu_i} - \chi_{\lambda+\mu_i}(a) e_{\mu_i} \bmod L_{i+1}^\lambda \equiv e_{\mu_i} \bmod L_{i+1}^\lambda$. Because L_i^λ is a free $U_q(\mathfrak{k}_-)$-module with basis $\{e_{\mu_k} | k \geqslant i\}$, there exist uniquely determined $u_{ij} \in U_q(\mathfrak{k}_-)$ such that $be_{\mu_i} = e_{\mu_i} + \sum_{j>i} u_{ij} e_{\mu_j}$ in N^{q^λ}.

Let $\mathfrak{p}$ be the vector space spanned by the vectors e_{μ_j} $j = 1, \ldots, 2p+1$. $U_q(\mathfrak{k}_-)\mathfrak{p}$ is an adjoint $U_q(\mathfrak{h})$-module; let $\{v_1, \ldots, v_m\}$ be a basis of the weight space $(U_q(\mathfrak{k}_-)\mathfrak{p})_{q^{\mu_i}}$ consisting of vectors of the form $e_{-\beta_r}^{k_r} \ldots e_{-\beta_1}^{k_1} e_{\mu_j}$ with $\mu_i = \mu_j - \sum_{l=1}^r k_l \beta_l$ and $v_1 = e_{\mu_i}$. In the same way, let $\{v_1^k, \ldots, v_{m_k}^k\}$ be a basis of $(U_q(\mathfrak{k}_-)\mathfrak{p})_{q^{\mu_i+\beta_k}}$.

For each $k = 1, \ldots, r$, there exist elements $p_{lj}^k \in U_q(\mathfrak{h})$ such that

$$e_{\beta_k} v_l \equiv \sum_{j=1}^{m_k} v_j^k p_{lj}^k \bmod U_q(\mathfrak{g})\mathfrak{k}_+.$$

Since the linear mapping $\varphi : U_q(\mathfrak{k}_-)\mathfrak{p} \otimes U_q(\mathfrak{h}) \to U_q(\mathfrak{g})$, $\varphi(\sum_l w_l \otimes u_l) = \sum_l w_l u_l$ is injective and $\mathrm{im}\varphi \cap U_q(\mathfrak{g})\mathfrak{k}_+ = \{0\}$, we see that for $q_1, \ldots, q_m \in U_q(\mathfrak{h})$

$$e_{\beta_k} \sum_{j=1}^{m} v_j q_j \equiv 0 \bmod U_q(\mathfrak{g})\mathfrak{k}_+ \tag{2}$$

for all $k = 1, \ldots, r$ if and only if

$$\begin{bmatrix} p^k_{11} & \cdots & p^k_{1m} \\ \vdots & & \vdots \\ p^k_{m_k 1} & \cdots & p^k_{m_k m} \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

for all $k = 1, \ldots, r$. Combining these equations together and doing some renumbering we obtain the following. There exist elements $p_{lj} \in U_q(\mathfrak{h})$ such that (2) holds if and only if

$$\begin{bmatrix} p_{11} & \cdots & p_{1m} \\ \vdots & & \vdots \\ p_{t1} & \cdots & p_{tm} \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{3}$$

where $t = \sum_{k=1}^{r} m_k$.

According to the first part of the proof we see that the equation

$$\begin{bmatrix} p_{11}(q^\lambda) & \cdots & p_{1m}(q^\lambda) \\ \vdots & & \vdots \\ p_{t1}(q^\lambda) & \cdots & p_{tm}(q^\lambda) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

has a unique solution $(\xi_1, \ldots, \xi_m) \in \mathbb{C}^m$ with $\xi_1 = 1$, if λ satisfies condition $(*)$, and so the rank of matrix $[p_{lj}(q^\lambda)]$ is $m - 1$ for all λ that satisfy $(*)$. The set of weights $\omega = q^\lambda$ for which λ does not satisfy $(*)$ is Zariski-closed in $(\mathbb{C}^*)^{n-1}$, and so the set of weights $\omega = q^\lambda$ for which λ satisfy $(*)$ is Zariski-dense. So the rank of the matrix $[p_{lj}]$ is $m-1$ and equation (3) has a solution $(q_1, \ldots, q_m)$ with $q_1 \neq 0$. Hence there exist $q_1, \ldots, q_m \in U_q(\mathfrak{h})$ with $q_1 \neq 0$ such that $e_{\beta_k} \sum_{j=1}^{m} v_j q_j \equiv 0 \bmod U_q(\mathfrak{g})\mathfrak{k}_+$ for all $k = 1, \ldots, r$.

Let p be an irreducible factor of q_1. If $p(q^\lambda) = 0$ for λ satisfying $(*)$, then by the uniqueness of the solution we get $q_j(q^\lambda) = 0$ for all j. Using Hilbert's *Nullstellensatz* we see that p is also a factor of each q_j, and so there is a solution of (3), such that $q_1(q^\lambda) \neq 0$ for all λ satisfying $(*)$. Now let $(q_1, \ldots, q_m)$ be a solution of (3) such that the q_i's have no common irreducible factors. Now we have proved the theorem with $u_i = q_1$. □

Let $S_q^0(\mathfrak{g}, \mathfrak{k}) \subset S_q(\mathfrak{g}, \mathfrak{k})$ be the subalgebra generated by s_{μ_i}'s and $U_q(\mathfrak{h})$.

Remark. In practice, the steps s_{μ_i} can be constructed by straightforward computation. For the quantum algebra pairs $U_q(\mathfrak{sl}(n-1)) \subset U_q(\mathfrak{sl}(n))$ it has been done in [6].

3.3. $\boldsymbol{U_q}(\mathfrak{k})$*-finite* $\boldsymbol{U_q}(\mathfrak{g})$*-modules*

Let $\Lambda \subset \mathfrak{h}^*$ be the set of integral weights, i.e. the weights for which $\langle \lambda | \alpha_j \rangle \in \mathbb{Z}$, $j = 2, \ldots, n$ and let Ω be the set of weights of the form $\omega = \epsilon \cdot q^\lambda$, $\lambda \in \Lambda$ and $\epsilon_i^4 = 1$. We define a partial ordering in Ω by setting $\epsilon \cdot q^\mu < \epsilon \cdot q^\lambda$ if the first non-zero element in the sequence $\langle \lambda - \mu | \alpha_2 \rangle, \ldots, \langle \lambda - \mu | \alpha_n \rangle$ is positive.

$U_q(\mathfrak{g})$-module V is $U_q(\mathfrak{k})$-finite if it is a sum of irreducible finite-dimensional $U_q(\mathfrak{k})$-modules. Let V^ω be the sum of all irreducible $U_q(\mathfrak{k})$-submodules with highest weight $\omega \in \Omega^+$. We denote by $V^+ \subset V$ the subspace which is annihilated by $\mathfrak{k}_+$ and $V^+_\omega = V^+ \cap V^\omega$.

Let D be the commutant of $U_q(\mathfrak{h})$ in $S_q^0(\mathfrak{g}, \mathfrak{k})$. Now V^+ is an $S_q(\mathfrak{g}, \mathfrak{k})$-module and V^+_ω a D-module in a natural way.

Theorem 3. If V is an irreducible $U_q(\mathfrak{k})$-finite $U_q(\mathfrak{g})$ module and $0 \neq v \in V^+$, then $V^+ = S_q^0(\mathfrak{g}, \mathfrak{k})v$.

Proof. Because V is irreducible we need to show that $V' = U_q(\mathfrak{k}_-)S_q^0(\mathfrak{g},\mathfrak{k})v$ is $U_q(\mathfrak{g})$-invariant. For this it is sufficient to show that for any $v' \in V'$, $e_{\mu_k}v' \in V'$ for $k = 1, \ldots, 2p+1$.

Let $v' = usv \in V'$, $u \in U_q(\mathfrak{k}_-)$, $s \in S_q^0(\mathfrak{g},\mathfrak{k})$. Using the commutation relations of Cartan–Weyl generators we see that

$$e_{\mu_k}v' = \sum_{i=1}^{2p+1} u_i e_{\mu_i} sv \qquad u_i \in U_q(\mathfrak{k}_-).$$

So we need only show that $e_{\mu_i}v' \in V'$ for any $v' \in S_q^0(\mathfrak{g},\mathfrak{k})v$. We do this by induction. Clearly we may take $s_{\mu_{2p+1}} = e_{\mu_{2p+1}}$ so

$$e_{\mu_{2p+1}}v' = s_{\mu_{2p+1}}v'.$$

Assume that $e_{\mu_i}v' \in V'$ for all $i > k$. Because $v' \in V^+$ we may assume that v' has weight q^λ with $\lambda \in \Lambda^+$.

If $\lambda + \mu_k \in \Lambda^+$, then by theorem 2

$$e_{\mu_k}u_k(q^\lambda)v' = s_{\mu_k}v' - \sum_{j>k} v_{jk}u_{jk}(q^\lambda)e_{\mu_j}v'$$

where $u_k(q^\lambda) \neq 0$ and $v_{jk} \in U_q(\mathfrak{k}_-)$; so $e_{\mu_k}v' \in V'$.

If $\lambda + \mu_k \notin \Lambda^+$, then

$$e_{\mu_k}v' \in \sum_{j>k} U_q(\mathfrak{k}_-)e_{\mu_j}v' = L_{k+1}.$$

Otherwise $e_{\mu_k}v' + L_{k+1}$ would be a highest weight vector of finite-dimensional $U_q(\mathfrak{k})$-module L_k/L_{k+1} with weight $q^{\lambda+\mu_k}$; a contradiction. □

For each $\omega \in \Omega^+$, let $M_\omega = \{u \in U_q(\mathfrak{g}) | uV_\omega^+ \subset V_{\omega'}^+$ for some $\omega' < \omega\}$. Denote $D_\omega = D/D \cap U_q(\mathfrak{g})M_\omega$. V^ω is a minimal component of a $U_q(\mathfrak{g})$-module V if $V^\omega \neq \{0\}$ and $V^{\omega'} = 0$ for all $\omega' < \omega$, $\omega' \in \Omega^+$. If V^ω is a minimal component of V, then V_ω^+ is a D_ω-module in a natural way. It follows from our choice of the partial ordering in Ω that any irreducible $U_q(\mathfrak{k})$-finite $U_q(\mathfrak{g})$-module has a unique minimal component.

In [6] we have proved the following theorem.

Theorem 4. The map $V \mapsto V_\omega^+$ gives a (1–1)-correspondence between the set $R(\omega)$ of equivalence classes of irreducible $U_q(\mathfrak{k})$-finite $U_q(\mathfrak{g})$-modules with a minimal component V^ω and the set $\mathrm{T}(\omega)$ of equivalence classes of irreducible D_ω-modules.

This theorem is useful when classifying the $U_q(\mathfrak{k})$-finite $U_q(\mathfrak{g})$-modules since it is usually quite simple to determine the structure of D_ω. In [6] we have done it in the $U_q(\mathfrak{sl}(n-1)) \subset U_q(\mathfrak{sl}(n))$ case. There we have used the explicit forms of the elements s_{μ_i}. However, that is probably not necessary, since methods analogous to those used in [11] can obviously also be used in the q-deformed case.

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